Analytic Solutions for Nonlocal Singlet-Even Nucleon-Nucleon Potentials*

LEE H. SCHICK
School of Physics, University of Minnesota, Minneapolis, Minnesota
(Received 12 August 1964)

The two-nucleon, singlet-even potential is assumed to be an integral operator with kernel

$$H'(\mathbf{r},\mathbf{r}')=u(r)H_c(\mathbf{r},\mathbf{r}')u(r')-[1-u(r)]V(r)\delta(\mathbf{r}-\mathbf{r}')[1-u(r')],$$

where $u(r)$ is a cutoff factor that goes to zero for large r and one for small r. In the region $u(r)=1$ an analytic solution to the Schrödinger equation with this potential is constructed for integral operators H_c which are functions of Hamiltonians $(\hbar^2/m)\nabla^2+U(r)$ whose eigenfunctions are known analytically. For cases where the Schrödinger equation with the potential $V(r)$ also admits an analytic solution, analytic expressions for the singlet-even phase shifts are derived. For the crude model with $u(r)$ square, $U(r)=V(r)$ square, the $^1S_0$ and $^1D_2$ phase shifts for energies up to 340 MeV are evaluated. The fits obtained to the experimental phase shifts are fair.

I. INTRODUCTION

THE purpose of this paper is to construct and apply to some preliminary calculations a phenomenological, nonlocal potential for singlet-even, nucleon-nucleon scattering at energies up to the order of 350 MeV, such that the resultant Schrödinger equation may be solved analytically. This potential is intended as an alternative to the hard-core potential,[1] the latter being mathematically difficult to use in calculating properties of complex nuclei. It is only for small interparticle distances that the nucleon-nucleon potential is taken to be of a nonlocal form for which the Schrödinger equation admits analytic solution. The longer range part of the nucleon-nucleon force is assumed to be describable in terms of some other (local) potential. Coulomb forces are neglected throughout the work.

Two classes of nonlocal potential have been used by other authors either for the entire nucleon-nucleon potential or as an alternative to a hard core. For one class (applied to nucleon-nucleon scattering by Moshinsky,[2] Werner,[3] Razavy, Field, and Levinger,[4] and Green[5]) the nonlocal part of the potential is represented by an explicit momentum-dependent term of the form

$$\lambda\mathbf{p}\cdot V(\mathbf{r})\mathbf{p}, \tag{1.1}$$

where $\mathbf{p}$ and $\mathbf{r}$ are the relative momentum and position of the nucleons respectively and $V(\mathbf{r})$ is a few-parameter shape. In the second class, the nonlocal part of the potential is taken to have the form of an integral operator; i.e., in the Schrödinger equation for the relative nucleon position the potential $V(\mathbf{r})$ times the wave function $\psi(\mathbf{r})$ is given by

$$V(\mathbf{r})\psi(\mathbf{r})=\int d\mathbf{r}\, V(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}'), \tag{1.2}$$

where the kernel $V(\mathbf{r},\mathbf{r}')$ does not contain a δ function of $\mathbf{r}-\mathbf{r}'$. It is this second class of nonlocal potential that is the subject of this paper.

Two types of nonlocal potential of the class given in Eq. (1.2) have been previously applied to the calculation of nucleon-nucleon singlet-even phase shifts. One of these is the nonlocal separable (NLS) potential

$$V(\mathbf{r},\mathbf{r}')=\lambda\textstyle\sum_l v_l(r)v_l(r')P_l(\hat{r},\hat{r}'), \tag{1.3}$$

such as introduced by Yamaguchi[6] and extended by Mitra and Narasimham.[7] Here $v_l(r)$ is a spherically symmetric shape for the lth angular momentum state and $P_l(\hat{r},\hat{r}')$ is the lth Legendre polynomial whose argument is the cosine of the angle between the unit vectors $\hat{r}=\mathbf{r}r^{-1}$ and $\hat{r}'=\mathbf{r}'(r')^{-1}$. The use of this potential reduces the problem of solving the Schrödinger equation to a quadrature for each partial wave. This potential, however, inherently contains a large number of parameters; i.e., each $v_l(r)$ contains (say) a range parameter and a strength parameter, while there is no *a priori* physical reason for relating the parameters in any one $v_l(r)$ to any other $v_{l'}(r)$ with $l\neq l'$.

In the other type of nonlocal potential of the class given in Eq. (1.2) the kernel has the form

$$V(\mathbf{r},\mathbf{r}')=\lambda u(r)G(|\mathbf{r}-\mathbf{r}'|)u(r'), \tag{1.4}$$

where $u(r)$ is a few-parameter spherically symmetric shape and $G(|\mathbf{r}-\mathbf{r}'|)$ is in effect a smeared-out δ function of $\mathbf{r}-\mathbf{r}'$. Such a potential contains only a small number of parameters, but in general it leads to a complicated Schrödinger equation which must be solved numerically. This type of potential by itself seems to have a further disadvantage. A few special cases of this potential form were used in a recent calculation by Giltinan and Thaler[8] who found it necessary to include a hard core in the shape $u(r)$ in order to fit both the $^1S_0$ and $^1D_2$ phase shifts for energies up to 310 MeV.[9]

* Supported in part by the U. S. Atomic Energy Commission.

[1] For calculations with a hard core, see J. L. Gammel, R. S. Christian, and R. M. Thaler, Phys. Rev. **105**, 311 (1957); J. L. Gammel and R. M. Thaler, *ibid.*, **107**, 291 (1957).

[2] M. Moshinsky, Phys. Rev. **106**, 117 (1957).

[3] E. Werner, Nucl. Phys. **35**, 324 (1962).

[4] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962).

[5] A. M. Green, Nucl. Phys. **33**, 218 (1962).

[6] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

[7] A. N. Mitra and V. L. Narasimham, Nucl. Phys. **14**, 407 (1960).

[8] D. A. Giltinan and R. M. Thaler, Phys. Rev. **131**, 805 (1963).

[9] For application of a nonlocal potential to nucleon-nucleus scattering, see F. Perey and B. Buck, Nucl. Phys. **32**, 353 (1962).

The present work seeks to combine the advantages of both the above types of integral operator potential—an inherently small number of parameters and ease of solution of the Schrödinger equation. The philosophy behind this work is not that the particular form of nonlocality used is physically meaningful in terms of a fundamental theory, although it appears that some form of nonlocal potential is needed,[8] but rather that a model of the nucleon-nucleon force may be obtained which can explain the two-body experimental data and so be a fit candidate for application to the few-nucleon, or many-nucleon, problem.

In the next section the desired analytic solution to the Schrödinger equation in the core region for a quite general type of nonlocal potential and some of the properties of this type of potential are discussed. In Sec. III the nucleon-nucleon singlet-even phase shift is calculated for a particular form of nonlocal core potential under the assumption that the long-range part of the Schrödinger equation also admits analytic solution. Some exploratory attempts to fit a particular model with square shapes to the experimental $^1S_0$ and $^1D_2$ phase shifts at energies up to 340 MeV are discussed and their results given in Sec. IV.

II. BASIC FORMALISM

In the center-of-mass coordinate system the Schrödinger equation for the scattering of two nucleons at an energy E is

$$H\psi_E=E\psi_E. \tag{2.1}$$

The Hamiltonian H involves the relative position $\mathbf{r}$ and the relative angular momentum $\mathbf{L}$ of the two nucleons, but, since application is to be made to singlet-even scattering, only even angular momentum states are of interest. The wave function ψ_E satisfies the usual scattering state boundary conditions; i.e., ψ_E is the sum of an incident plane wave and a sacttered wave which for large $r=|\mathbf{r}|$ has the form of a scattering amplitude times an outgoing spherical wave.

The Hamiltonian H is taken to have the form

$$H=K+H', \tag{2.2}$$

where K is the relative kinetic energy operator and H' is an integral operator such that

$$H'\psi_E(\mathbf{r})=\int d\mathbf{r}\, H'(\mathbf{r},\mathbf{r}')\psi_E(\mathbf{r}'). \tag{2.3}$$

The kernel $H'(\mathbf{r},\mathbf{r}')$ is taken to have the form

$$H'(\mathbf{r},\mathbf{r}')=u(r)H_c(\mathbf{r},\mathbf{r}')u(r') +[1-u(r)]H_1(\mathbf{r},\mathbf{r}')[1-u(r')], \tag{2.4}$$

where

$$u(r)\to\begin{cases}1, & r\ll c\\ 0, & r\gg c\end{cases}. \tag{2.5}$$

The kernel $H_c(\mathbf{r},\mathbf{r}')$ represents the nucleon-nucleon interaction deep inside the core, $H_1(\mathbf{r},\mathbf{r}')$ represents the long-range part of this interaction, and $u(r)$ is a radial cutoff factor with c being in some sense the core radius. Such a model is based on the interpretation of the fact that the $^1S_0$ phase shift becomes negative at high energies while the $^1D_2$ phase shift does not[10] as due to a repulsive core which strongly influences the S-wave part of the wave function, but because of its short range, influences the D wave only weakly, if at all.

The Schrödinger equation may now be written symbolically as

$$[K+uH_cu+(1-u)H_1(1-u)]\psi_E=E\psi_E. \tag{2.6}$$

It is assumed that H_1 is describable in terms of a local potential,

$$H_1(\mathbf{r},\mathbf{r}')\propto V(r)\delta(\mathbf{r}-\mathbf{r}'), \tag{2.7}$$

so that methods of handling Eq. (2.6) for $r\gg c$, where

$$[K+H_1]\psi_E=E\psi_E \tag{2.8}$$

are well known, while H_c is assumed to include a nonlocal nonseparable potential. It is for the region inside the core, where Eq. (2.6) becomes

$$[K+H_c]\psi_E=E\psi_E, \tag{2.9}$$

that an analytically solvable model is to be investigated.

The basis of the development given below is the assumption that $K+H_c$ has the form

$$K+H_c=K+U+H_c'=H_0+H_c', \tag{2.10}$$

where U is a local potential[11] such that the functions ϕ_ϵ that satisfy

$$H_0\phi_\epsilon=\epsilon\phi_\epsilon \tag{2.11}$$

are known analytically for all r, and H_c' which represents the nonlocal potential is itself a function of H_0,

$$H_c'=f(H_0). \tag{2.12}$$

But H_c' is an integral operator, so $f(H_0)$ must contain an inverse differential operator $[f_1(H_0)]^{-1}$; i.e., H_c' may be written

$$H_c'=[f_1(H_0)]^{-1}f_2(H_0) \tag{2.13}$$

and the $f_j(H_0)$, $j=1, 2$, contain only positive powers of H_0. The substitution of Eqs. (2.10) and (2.13) into Eq. (2.9) yields the integrodifferential equation

$$\{H_0+[f_1(H_0)]^{-1}f_2(H_0)\}\psi_E=E\psi_E, \quad r\ll c. \tag{2.14}$$

This equation may be converted into a differential equation by operation from the left with $f_1(H_0)$, which

[10] G. Breit, M. H. Hull, K. E. Lassila, and K. D. Pyatt, Phys. Rev. **120**, 2227 (1960).

[11] Outside the spirit of this work U and H_1 could be taken to be NLS potentials. In such a case the results of this section and the next with slight modification would also be applicable.

yields after rearrangement

$$\{f_1(H_0)[H_0-E]-f_2(H_0)\}\psi_E=0. \quad (2.15)$$

The general solution to Eq. (2.15) may now be constructed by application of the well-known fact that for a function $f(H_0)$, Eq. (2.11) implies

$$f(H_0)\phi_\epsilon=f(\epsilon)\phi_\epsilon. \quad (2.16)$$

In particular, the expansion of ψ_E in terms of the ϕ_ϵ,

$$\psi_E=\sum_\epsilon A(E,\epsilon)\phi_\epsilon \quad (2.17)$$

when substituted into Eq. (2.15) yields

$$\sum_\epsilon A(E,\epsilon)\{f_1(\epsilon)[\epsilon-E]-f_2(\epsilon)\}\phi_\epsilon=0. \quad (2.18)$$

In general,

$$f_1(\epsilon)[\epsilon-E]-f_2(\epsilon)=0 \quad (2.19)$$

has a set of solutions

$$\epsilon=\epsilon_1, \epsilon_2, \cdots,$$

so that the left-hand side of Eq. (2.18) can be made to vanish by the requirement

$$A(E,\epsilon)=0, \quad \epsilon\neq\epsilon_1, \epsilon_2, \cdots; \quad (2.20)$$

i.e., ψ_E, as given by Eq. (2.17) but with the sum running only over the solutions to Eq. (2.19), is the general solution of Eq. (2.15).

As Eq. (2.15) was in effect obtained from Eq. (2.14) by differentiation, the desired solution to Eq. (2.14) may be obtained from the ψ_E just constructed. First, only functions ϕ_ϵ which satisfy the same boundary condition at $r=0$ as the desired solution to Eq. (2.14) are to be included in the expansion of ψ_E. Second, the so far arbitrary coefficients $A(E,\epsilon)$ may be determined by the requirement that when ψ_E is substituted into Eq. (2.14) written out in full (not merely in symbolic form) an identity results. This requirement yields a sufficient number of relations among the coefficients $A(E,\epsilon)$ to determine them all in terms of the value of the logarithmic radial derivative of the wave function at (say) $r=r_0<c$ and the normalization of the wave function which, of course, is quite arbitrary since it plays no role in the scattering problem. This completes the construction of the desired solution to Eq. (2.14) for a given $f(H_0)$.

The integral operator $f(H_0)$ is completely arbitrary except for the symmetries imposed by the usual invariance requirements.[12] These requirements are satisfied if the kernel $H_c(\mathbf{r},\mathbf{r}')$ of this operator is a real symmetric function of its arguments. A natural choice for such an operator is a Green's function of H_0,

$$f(H_0)\propto[\mathcal{E}-H_0]^{-1}=[\mathcal{E}-K-U]^{-1}. \quad (2.21)$$

To ensure the reality of $H_c(\mathbf{r},\mathbf{r}')$, that is to avoid the addition of a small imaginary term to the denominator of $f(H_0)$, $\mathcal{E}$ is chosen to satisfy

$$\mathcal{E}<U(r), \quad \text{all } r. \quad (2.22)$$

In other words, with this condition, $f(H_0)$ operating on any eigenstate ϕ_ϵ of H_0 gives a finite result; for a repulsive potential, $U(r)\geq 0$, H_0 has only positive eigenvalues, while for an attractive potential, $U_0\leq U(r)\leq 0$, $\mathcal{E}<U_0$ implies $\mathcal{E}<\epsilon_b$, where ϵ_b is the energy of the lowest bound state of $U(r)$.[13] Other choices for $f(H_0)$ include derivatives of this Green's function with respect to $\mathcal{E}$ or any such function multiplied by a positive integral power of H_0. In all of these, $\mathcal{E}$ is one of the free parameters of the model to be determined by comparison with experiment.

The form given in Eq. (2.21) for $f(H_0)$, or in any of the other choices mentioned above, does not define $f(H_0)$ completely. There are two boundary conditions on the kernel $H_c(\mathbf{r},\mathbf{r}')$ as a function of $\mathbf{r}$ yet to be specified. One of these is the usual condition that $H_c(\mathbf{r},\mathbf{r}')$ is well behaved at $r=0$. Usually (e.g., in the treatment of the free wave Green's function in a two-body scattering or bound-state problem) the other boundary condition imposed on a Green's function relates to the behavior of its kernel as $r\to\infty$. In the case under discussion the region of interest is $r<c$ so that this second usual boundary condition may or may not be applicable. If the cutoff factor $u(r)$ is taken to be sharp, $u(r)=0$ for $r>c$, then there is no physical basis for the preference of one type of behavior of $H_c(\mathbf{r},\mathbf{r}')$ at very large r over some other type of behavior. If, on the other hand, $u(r)$ is smooth then a boundary condition on $H_c(\mathbf{r},\mathbf{r}')$ as $r\to\infty$ is applicable. In fact if this boundary condition is taken to be that $H_c(\mathbf{r},\mathbf{r}')$ vanishes rapidly as $r\to\infty$, then the self-consistency of the model in that H_1 alone represents the long-range part of the potential is guaranteed. This same boundary condition may be carried over to the case of a sharp cutoff by the consideration that a sharp cutoff is merely a convenient approximation to the more physically reasonable smooth cutoff.

The final point to be investigated is whether $uf(H_0)u$ is attractive, or repulsive, or in fact either. If H_0 consists only of the relative kinetic energy operator $U(r)=0$, then for the types of $f(H_0)$ discussed above $uf(H_0)u$ is either a positive definite or a negative definite operator; i.e., $uf(H_0)u$ represents either a repulsive or an attractive force. For example, with

$$f(H_0)=-NH_0[\mathcal{E}-H_0]^{-1}, \quad (2.23)$$

where

$$H_0=-(\hbar^2/2\mu)\nabla^2, \quad \mathcal{E}<0, \quad (2.24)$$

and the expansion for arbitrary

$$u(r)\psi(\mathbf{r})=(2\pi)^{-3/2}\int d\mathbf{q}a(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (2.25)$$

[12] For a discussion of these requirements, see S. Okubo and R. Marshak, Ann. Phys. (N. Y.) 4, 166 (1958).

[13] If $U(r)$ is singular, a principal value integral may be used and condition (22) avoided.

the matrix element

$$(\psi,uf(H_0)u\psi)$$

is easily seen to be equal to

$$N\int d\mathbf{q}\,|\,a(\mathbf{q})\,|^2(\hbar^2q^2/2\mu)[\,|\,\mathcal{E}\,|+(\hbar^2/2\mu)q^2]^{-1},$$

an expression whose sign is that of the parameter N. For N positive (negative) $uf(H_0)u$ is repulsive (attractive).

If H_0 contains a repulsive potential or an attractive potential that is too weak to support even an S-wave bound state, $uf(H_0)u$ is again a definite operator. However, if H_0 contains an attractive potential which is strong enough to support a bound state in (say) the lth angular momentum channel, then a different situation exists. In this case the lth partial-wave part of $(\psi,ufu\psi)$ contains a contribution from the bound state which may not have the same sign as the contributions from the positive energy states. For example, if $f(H_0)$ has the form given in Eq. (2.23) but with

$$H_0=-(\hbar^2/2\mu)\nabla^2-U(r), \tag{2.26}$$

where $U(r)$ satisfies

$$0\leq U(r)\leq U_0,\quad 0\leq r\leq\infty \tag{2.27}$$

and has a bound state in the lth angular momentum channel at an energy $-\epsilon$, then the lth partial-wave part of $(\psi,uf(H_0)u\psi)$ has the form

$$\frac{-\epsilon N\,|\,a_l(-\epsilon)\,|^2}{|\,\mathcal{E}\,|-\epsilon}+(\text{positive energy terms}), \tag{2.28}$$

where

$$a_l(-\epsilon)=(\phi_{-\epsilon,l},u\psi)$$

and $\phi_{-\epsilon,l}$ is the bound-state wave function. The sign of the positive energy contribution to expression (2.28) is as before the same as the sign of N, whereas, since $|\,\mathcal{E}\,|>U_0$ [see Eqs. (2.22), (2.26), (2.27)] so that $|\,\mathcal{E}\,|>\epsilon$, the sign of the explicit term in expression (2.28) is opposite that of N. If the parameters $\mathcal{E}$, N, etc., are chosen so that the positive energy terms dominate expression (2.28), the situation is the same as that described above; i.e., $uf(H_0)u$ is attractive or repulsive as N is negative or positive for all angular momentum channels including the lth. If the pole term dominates then $uf(H_0)u$ is attractive (repulsive) in the lth angular momentum channel while being repulsive (attractive) for all other angular momenta. In the next section it is assumed that the positive energy terms are dominant.

With an analytic solution to Eq. (2.16) deep inside the core, it remains only to assume that $u(r)$ is square

$$\begin{aligned} u(r)&=1,\quad r\leq c\\ &=0,\quad r>c\end{aligned} \tag{2.29}$$

and H_1 is a potential such that Eq. (2.8) admits an analytic solution, to obtain a Schrödinger equation which is analytically solvable everywhere. On the basis of meson theory this last assumption is not correct as the one-pion-exchange potential (OPEP) is not a potential for which the Schrödinger equation may be solved analytically. Nevertheless, for the purpose of a detailed description of the model under consideration it is useful to make this assumption. For a more realistic attempt to fit the scattering data than is carried out below, the wave function outside the core may be described numerically.

III. DETAILED EXAMPLE

In order to exhibit the details of the formalism a particular example is now investigated. The various operators defined above are chosen to have the forms

$$H_0=-\nabla^2-U, \tag{3.1}$$

$$H_1=-V, \tag{3.2}$$

$$\mathcal{E}=-\xi^2<0, \tag{3.3}$$

with $f(H_0)$ given by Eq. (2.23). The right-hand sides of Eqs. (3.1), (3.2), (3.3) are in units of $(\hbar^2/m)$ where m is the nucleon mass. The operators U and V are taken to be local positive potentials so that $-U$ and $-V$ are attractive. The cutoff $u(r)$ is chosen to be given by Eq. (2.24).

Symbolically the Schrödinger equation is

$$\begin{aligned}(\nabla^2+k^2)\psi=\{&-(1-u)V(1-u)\\&+u[-U+N(\nabla^2+U-\xi^2)^{-1}(\nabla^2+U)]u\}\psi,\end{aligned} \tag{3.4}$$

where $k^2=(mE/\hbar^2)$. The application of the identity

$$(\nabla^2+U-\xi^2)^{-1}(\nabla^2+U)=1+\xi^2(\nabla^2+U-\xi^2)^{-1} \tag{3.5}$$

to Eq. (3.4) and a partial-wave analysis yields the equation for the lth partial wave

$$\begin{aligned}(D_l^2+k^2)\psi_l(r)=&-[1-u(r)]^2V(r)\psi_l(r)\\&-u(r)[U(r)-N]u(r)\psi_l(r)+N\xi^2u(r)\\&\times\int_0^\infty(r')^2dr'G_l(\xi^2;r,r')u(r')\psi_l(r'),\end{aligned} \tag{3.6}$$

where

$$D_l^2=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{r^2}, \tag{3.7}$$

and $G_l(\xi^2;r,r')$ satisfies

$$[D_l^2+U(r)-\xi^2]G_l(\xi^2;r,r')=(rr')^{-1}\delta(r-r'). \tag{3.8}$$

The boundary conditions on $\psi_l(r)$ are

$$\psi_l(r)\text{ is regular at }r=0, \tag{3.9}$$

$$\psi_l(r)\to j_l(kr)+f_lr^{-1}e^{ikr}\quad\text{as}\quad r\to\infty, \tag{3.10}$$

$$\psi_l(r)\text{ is continuous at }r=c, \tag{3.11}$$

$$d\psi_l(r)/dr \text{ is continuous at } r=c. \tag{3.12}$$

In Eq. (3.10), $j_l(kr)$ is the lth spherical Bessel function[14] and f_l is the lth scattering amplitude

$$f_l=k^{-1}e^{i\delta_l}\sin\delta_l, \tag{3.13}$$

where δ_l is the lth phase shift. Equations (3.9) and (3.10) are the usual scattering state boundary conditions for the lth partial wave. The boundary conditions at $r=c$ follow from the form of Eqs. (3.6) and (3.8); i.e., from Eq. (3.8), $G_l(\xi^2;r,r')$ is a continuous function of r (although its derivative is discontinuous at $r=r'$) so that if $\psi_l(r)$, or its derivative, has a finite jump discontinuity at $r=c$ the right side of Eq. (3.6) has at worst a finite jump discontinuity at $r=c$ while the left-hand side of this equation has an infinite singularity at this point.

To guarantee that the integrand is well behaved at $r'=0$, the boundary condition (3.9) is also imposed on $G_l(\xi^2;r,r')$. The second boundary condition on $G_l(\xi^2;r,r')$ —whose specification completes the definition of this kernel—is for the moment left arbitrary. This is a boundary condition at $r=c$ and will be referred to below as boundary condition A.

In the region $r>c$ Eq. (3.6) becomes

$$[D_l^2+V(r)+k^2]\psi_l(r)=0. \tag{3.14}$$

For convenience it is assumed that $V(r)=0$ for $r>b$. Furthermore, the functions $\chi_{l1}(\alpha^2,r)$, $\chi_{l2}(\alpha^2,r)$ which satisfy

$$[D_l^2+V(r)+\alpha^2]\chi_l(\alpha^2,r)=0 \tag{3.15}$$

[where the subscript 1 denotes that solution to Eq. (3.15) which is regular at $r=0$ and the subscript 2 denotes a solution to Eq. (3.15) that is irregular at $r=0$, so that χ_{l1} and χ_{l2} are linearly independent] are assumed to be known. The desired solution to Eq. (3.14) may be written

$$\begin{aligned}\psi_l(r)&=B_l[\chi_{l1}(k^2,r)+C_l\chi_{l2}(k^2,r)], \quad c<r\leq b\\ &=i^l[j_l(kr)+ikf_lh_l^{(1)}(kr)], \quad r>b,\end{aligned} \tag{3.16}$$

where $h_l^{(1)}(kr)$ is the spherical Hankel function of the first kind[14] and B_l, C_l are constants to be determined from the boundary conditions on $\psi_l(r)$.

For $r\leq r_c$ Eq. (3.6) becomes

$$[D_l^2+U(r)-N+k^2]\psi_l(r)$$
$$=N\xi^2\int_0^c(r')^2dr'G_l(\xi^2;r,r')\psi_l(r'). \tag{3.17}$$

From the discussion in Sec. II, it follows that the solution to this equation has the form

$$\psi_l(r)=\sum A_l(k^2,\beta^2)\phi_{l1}(\beta^2,r), \tag{3.18}$$

where $\phi_{l1}(\beta^2,r)$ satisfies

$$[D_l^2+U(r)+\beta^2]\phi_{l1}(\beta^2,r)=0 \tag{3.19}$$

and is regular at the origin, while the sum runs over those values of β^2 that satisfy

$$\beta^2+(\beta^2+\xi^2)^{-1}N\beta^2-k^2=0. \tag{3.20}$$

In full then

$$\psi_l(r)=A_l(k^2,\beta_1^2)\phi_{l1}(\beta_1^2,r)+A_l(k^2,\beta_2^2)\phi_{l1}(\beta_2^2,r), \tag{3.21}$$

where

$$\beta_j^2=\tfrac{1}{2}(k^2-\xi^2-N)-\tfrac{1}{2}(-1)^j \times[(k^2-\xi^2-N)^2+4\xi^2k^2]^{1/2}, \tag{3.22}$$

for $j=1, 2$.

To determine the ratio of the coefficients in Eq. (3.21) it is necessary to substitute $\psi_l(r)$ as given by this equation into Eq. (3.17), and evaluate both sides of the result explicitly. The kernel $G_l(\xi^2;r,r')$ in the region $r,r'<c$ may be shown to be given by[15]

$$G_l(\xi^2,r,r')=\lambda_l\phi_{l1}(-\xi^2,r_<)\phi_{l3}(-\xi^2,r_>), \tag{3.23}$$

where $r_<(r_>)$ is the lesser (greater) of r, r'; $\phi_{l1}(-\xi^2,r)$, $\phi_{l3}(-\xi^2,r)$ are solutions to Eq. (3.19) with $\beta^2=-\xi^2$ such that ϕ_{l1} is regular at $r=0$ and ϕ_{l3} satisfies boundary condition A; λ_l is given by

$$\lambda_l=(r')^2\Delta_l(1,3;-\xi^2,-\xi^2;r'), \tag{3.24}$$

with the Wronskian Δ_l defined as

$$\Delta_l(i,j;\lambda^2,\gamma^2;x)=\phi_{li}(\lambda^2,x)\phi_{lj}'(\gamma^2,x) -\phi_{li}'(\lambda^2,x)\phi_{lj}(\gamma^2,x); \tag{3.25}$$

and a prime denotes differentiation with respect to x. With the use of these relations and the identity [easily derived from Eq. (3.19)]

$$\int_x^y(r')^2\phi_{li}(\lambda^2,r')\phi_{lj}(\gamma^2,r')dr' =(\lambda^2-\gamma^2)^{-1}[y^2\Delta_l(i,j;\lambda^2,\gamma^2;y) -x^2\Delta_l(i,j;\lambda^2,\gamma^2;x)] \tag{3.26}$$

Eq. (3.21) when substituted into Eq. (3.17) gives[16]

$$\sum_{i=1}^2 A_l(k^2,\beta_i^2)[-\beta_i^2+k^2-N+(\beta_i^2+\xi^2)^{-1}N\xi^2] \times\phi_{l1}(\beta_i^2,r)=N\xi^2c^2\lambda_l\phi_{l1}(-\xi^2,r)\sum_{i=1}^2 A_l(k^2,\beta_i^2) \times(\beta_i^2+\xi^2)^{-1}\Delta_l(1,3;\beta_i^2,-\xi^2;c). \tag{3.27}$$

[14] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., pp. 77–79.

[15] See P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, pp. 791–895; or R. G. Newton, J. Math. Phys. **1**, 319 (1960).

[16] For potentials no more singular at the origin than r^{-1}, the contribution to the right-hand side of Eq. (3.26) from the lower limit $x=0$ vanishes.

By virtue of Eq. (3.20) the left-hand side of this equation vanishes identically so the relation between the A_l's that must hold for Eq. (3.27) to be an identity is

$$\sum_{i=1}^{2} A_l(k^2,\beta_i^2)(\beta_i^2+\xi^2)^{-1}\Delta_l(1,3:\beta_i^2-\xi^2;c)=0. \quad (3.28)$$

The boundary conditions on $\psi_l(r)$ at $r=b$, c may now be used to obtain an expression for the phase shift δ_l. From Eq. (3.16) the continuity of the logarithmic derivative of $\psi_l(r)$ at $r=b$ yields

$$k\cot\delta_l=[N_{l1}+C_lN_{l2}][D_{l1}+C_lD_{l2}]^{-1}, \quad (3.29)$$

where

$$N_{lj}=k[n_l'(kb)\chi_{lj}(k^2,b)-n_l(kb)\chi_{lj}'(k^2,b)], \quad j=1,2, \quad (3.30)$$

$$D_{lj}=j_l'(kb)\chi_{lj}(k^2,b)-j_l(kb)\chi_{lj}'(k^2,b), \quad j=1,2, \quad (3.31)$$

while the continuity of this same derivative at $r=c$ gives from Eqs. (3.16), (3.21), (3.28)

$$C_l=\frac{-(\beta_1^2+\xi^2)X_l(\beta_2^2,\beta_1^2)+(\beta_2^2+\xi^2)X_l(\beta_1^2,\beta_2^2)}{(\beta_1^2+\xi^2)Y_l(\beta_2^2,\beta_1^2)+(\beta_2^2+\xi^2)Y_l(\beta_1^2,\beta_2^2)}, \quad (3.32)$$

where

$$X_l(\lambda^2,\gamma^2)=\Delta_l(1,3;\lambda^2,-\xi^2;c)\times\tilde{\Delta}_l(1,1;\gamma^2,k^2;c), \quad (3.33)$$

$$Y_l(\lambda^2,\gamma^2)=\Delta_l(1,3;\lambda^2,-\xi^2;c)\times\tilde{\Delta}_l(1,2;\gamma^2,k^2;c),$$

$$\tilde{\Delta}_l(i,j;\lambda^2,\gamma^2;x)=\phi_{li}(\lambda^2,x)\chi_{lj}'(\gamma^2,x)-\phi_{li}'(\lambda^2,x)\chi_{lj}(\gamma^2,x), \quad (3.34)$$

and a prime again denotes differentiation with respect to x.

Equations (3.29) through (3.34) give the phase shift δ_l in terms of the parameters of the model used. If $U(r)$, $V(r)$ are taken to be two-parameter potentials with U_0, V_0 being depth parameters and r_1, r_2 being range parameters of $U(r)$, $V(r)$ respectively, then the model contains seven free parameters: U_0, V_0, r_1, r_2, N, ξ^2, c. [If $V(r)$ is a square well, $b=r_2$; otherwise the introduction of b is just a device that allows the wave function to be written down in the region of very large r independent of the shape of $V(r)$.] This is too many parameters for a fit of the singlet-even scattering data to be very meaningful. There are of course many ways of reducing the number of free parameters. For example V_0 and r_2 may be fixed by the choice of the OPEP for $V(r)$. In this case the wave functions $\chi_{lj}(k^2,r)$ must be found numerically. Again, two free parameters may be eliminated by the choice $U(r)=0$, or $U(r)=V(r)$, or $r_1=r_2$ and $\xi^2=N$, although these latter choices tend to obscure the dependence of the phase shift on particular features of the model. In any case the number of free parameters may be reduced to at most five, which is only one more than the number used in, for example, Refs. 4 and 8. These five parameters could be chosen so that $k\cot\delta_l$ as given above yields the correct $^1S_0$ scattering length, effective range, and phase shift at (say) 350 MeV as well as the correct $^1D_2$ phase shifts at (say) 100 and 350 MeV. The test of the model would then be the fit obtained to the $^1S_0$ and $^1D_2$ (and possibly $^1G_4$) phase shifts at intermediate energies.

Except for δ-function potentials and the like the choice of a local potential for which the eigenfunctions of Eq. (3.19) are known analytically for all angular momenta is limited to a square shape. However, there are a variety of local potentials (e.g., exponential, Bargmann potentials,[17] Yost potential[18]) for which the S-wave eigenfunctions are so known. For such potentials the S-wave part of the core may be treated exactly while, since the $^1D_2$ phase shift is small for energies below 350 MeV, a perturbation treatment of the entire potential should suffice for the D wave.

IV. NUMERICAL APPLICATION

The numerical analysis described below was of a strictly exploratory nature, rather than an attempt to fit the data with a model containing a potential shape justifiable on a fundamental basis. The computations were performed on the CDC 1604 at the University of Minnesota Numerical Analysis Center. A straightforward input-output program was used so that the results obtained are in no sense a "best fit."

Two types of numerical computations were performed. In both of these it was assumed that $u(r)$ was square shaped and that

$$V(r)=U(r)=U_0\begin{cases}1, & r\leq b\\ 0, & r>b.\end{cases} \quad (4.1)$$

In the first type of computation the further assumption was made that at low energies the core plays no role; i.e., the expansion of $k\cot\delta_0$ through order k^2 is the same as it would be in the case $N=0$. This assumption leads to the single condition[19]

$$\Delta_0(1,3;0,-\xi^2,c)=0, \quad (4.2)$$

which reduces the total number of free parameters to four, of which only two—U_0 and b—are available to fit the low-energy $^1S_0$ scattering data. This condition cannot be satisfied if boundary condition A is chosen to correspond to a rapid vanishing of $G_0(\xi^2;r,r')$ at large r; i.e., if

$$\phi_{0,3}(-\xi^2,r)\to r^{-1}\exp[-(\xi^2-U_0)^{1/2}r], \quad r>r'. \quad (4.3)$$

[17] V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949).
[18] R. Jost, Helv. Phys. Acta **20**, 256 (1947).
[19] See Appendix.

TABLE I. Sets of parameters for which phase shifts are given in Tables II–IV.

Set number	c(F)	b(F)	U_0(F^{-2})	ξ^2(F^{-2})	N(F^{-2})
1	2.059	2.540	0.352	0.398	1.230
2	1.666	2.540	0.352	0.469	3.585
3	1.182	2.540	0.352	0.845	490.0
4	1.0	1.5	3.5086	4.00	150
5	0.9	1.4	3.8264	4.00	400
6	0.9	1.4	5.0295	6.25	2350

For the sake of simplicity then, a rising exponential behavior was chosen for $\phi_{03}(-\xi^2,r)$, the absence of the nonlocal potential at distances greater than c now being due solely to the cutoff factor $u(r)$. With this choice Eq. (4.2) reduces to

$$\xi c=(U_0c^2)^{1/2}\operatorname{cosec}(U_0c^2)^{1/2}. \tag{4.4}$$

The values used for U_0 and b—chosen to give a $^1S_0$ scattering length and effective range of -23.74 and 2.65 F, respectively[20]—were $U_0=0.352$ F^{-2} and $b=2.54$ F. Various combinations of N and c, with ξ given by Eq. (4.4) for each value of c, were then used to obtain a fit to the $^1S_0$ phase shift at 340 MeV. The $^1S_0$ phase shift at intermediate energies as well as the $^1D_2$ phase shift from 100 to 340 MeV were also calculated for each such fit.

TABLE II. The ^1S phase shifts from the first type of computation.[a]

E (MeV)	20	100	180	260	340
$\delta_0(1)$	0.885	0.304	0.047	−0.098	−0.195
$\delta_0(2)$	0.887	0.326	0.089	−0.061	−0.195
$\delta_0(3)$	0.888	0.333	0.103	−0.047	−0.195
Breit's K_0(YLAM)	0.856	0.380	0.136	−0.033	−0.195

[a] Phase shifts are in radians. $\delta_0(i)$ is the phase shift calculated for the ith set of parameters given in Table I. For all three $\delta_0(i)$, the scattering length $a=-23.74$ F and the effective range $r=2.65$ F.

The results for the $^1S_0$ phase shifts for three different core ranges, as well as the values from the Breit (YLAM) phase-shift analysis,[10] are given in Table II. (See Table I for the corresponding three full sets of parameters—sets 1, 2, 3—as well as the sets of parameters—sets 4, 5, 6—used in the second type of computation.) It is clear from this table that the fit to the data obtained here is acceptable only when the nonlocal potential is a core potential; i.e., $c\lesssim(b/2)$. The use of core radii $c\lesssim 1.0$ F requires such large values of N, ($N\gtrsim 10^5$), to drive the $^1S_0$ phase shift negative at 340 MeV that detailed calculations were not performed in this region.

The results for the $^1D_2$ phase shift (not shown) were very poor, being far too attractive at lower energies ($\approx$100–200 MeV) and very much too repulsive at higher energies. Here again, though, the smaller core radii gave the better results. A possible cause of the failure of the model to fit the $^1D_2$ phase shift is the distortion introduced by the use of an unphysical square shape for $U(r)$; i.e., the very long-range attraction of this shape necessitates the use of such a large N to drive the $^1S_0$ phase shift negative at high energies that the $^1D_2$ phase shift is also driven negative at these energies.[21]

In the second type of computation Eq. (4.2) was not used. The large r behavior of $\phi_{03}(-\xi^2,r)$ was chosen to be that given in Eq. (4.3); otherwise the model used was the same as that used above. Here five free parameters were available.

TABLE III. The ^1S phase shifts from the second type of computation.[a]

E (MeV)	20	100	180	260	340
$\delta_0(4)$	1.005	0.472	0.150	−0.083	−0.259
$\delta_0(5)$	1.039	0.544	0.233	0.012	−0.165
$\delta_0(6)$	1.012	0.485	0.154	−0.087	−0.279
Breit's K_0(YLAM)	0.856	0.380	0.136	−0.033	−0.195
Gammel-Thaler	0.859	0.379	0.120	−0.069	···
RFL[b]	0.852	0.341	0.103	−0.079	−0.192

[a] Phase shifts are in radians. $\delta_0(i)$ is the phase shift calculated for the ith set of parameters in Table I. The scattering length a and effective range r are (in F) as follows; $a(4)=-24.2$, $a(5)=-23.9$, $a(6)=a(G-T)=-23.6$ $a(\text{RFL})=\infty$, $r(4)=2.07$, $r(5)=1.92$, $r(6)=2.03$, $r(G-T)=2.65$, $r(\text{RFL})=2.71$.
[b] Reference 4.

As a starting point the value $\xi^2=4.00$ F^{-2} (i.e., a nonlocal range, ξ^{-1}, of half a fermi) was chosen. For each of the sets $c=0.5$ and 1.0 F, $b=0.5$ to 2.5 F (in steps of 0.5 F with $b\geq c$), U_0 was varied over the range 0.14 to 3.98 F^{-2} and N was varied from 1.0 to 10^4 F^{-2}. The $^1S_0$ and $^1D_2$ phase shifts were calculated at energies up to 340 MeV for each set of values assigned to c, b, U_0, and N and the results compared with the Breit (YLAM) phase-shift analysis.[10] A rough fit to both phase shifts over the entire energy range was found at $c=1.0$ F, $b=1.5$ F. These parameters were kept fixed while U_0 and N were varied in more detail until the number 4 set of results in Tables III and IV was obtained.

TABLE IV. The ^1D phase shifts from the second type of computation.[a]

E (MeV)	100	180	260	340
$\delta_2(4)$	0.100	0.143	0.153	0.129
$\delta_2(5)$	0.097	0.135	0.156	0.141
$\delta_2(6)$	0.060	0.106	0.138	0.146
Breit's K_2(YLAM)	0.072	0.120	0.160	0.184
Gammel-Thaler	0.096	0.181	0.239	···
RFL[b]	0.114	0.200	0.246	0.259

[a] Phase shifts are in radians. $\delta_2(i)$ is the phase shift calculated for the ith set of parameters in Table I.
[b] Reference 4.

[20] J. L. Gammel and R. M. Thaler, *Progress in Elementary Particle and Cosmic-Ray Physics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. V, p. 156.

[21] Spurious effects due to the very high momentum components involved in the sharp corner of the square well may also be present.

The ranges c and b were then varied separately in steps of 0.1 F with U_0 and N adjusted accordingly to maintain, or improve, if possible the fair fit already obtained. It was found, for example, that an increase (decrease) in b with the approximate S-wave fit maintained, caused the D-wave phase shift to become too attractive (repulsive). An increase (decrease) in c had results similar to a decrease (increase) in b. These variations in the phase shift are of course just what would be expected.

The parameters c and b were then decreased simultaneously to the respective values of 0.9 and 1.4 F, and the parameters U_0 and N were varied until the number 5 set of results listed in Tables III and IV was obtained. The D-wave fit here is better but the S-wave fit slightly worse than with the number 4 set of parameters. Corresponding results (not shown) were obtained with $c=1.1$ F and $b=1.6$ F.

Further attempts to find a significantly better fit to the $^1S_0$ phase shift than those given in Table III without disturbing the good D-wave fit—attempts which used values of c in the range 0.8 to 1.4 F and values of b from 0.8 to 2.0 F, both in steps of 0.1 F with $b \geq c$—were not successful. In particular the $^1S_0$ effective range could only be fit at the expense of the high-energy $^1S_0$ phase shift. It should again be noted that the fits obtained here, rough as they are, were obtained with the nonlocal potential again acting as a core potential (i.e., $c \approx 2b/3$), but with the core radius a large enough fraction of b to make the D-wave phase shift quite sensitive to the value of c.

Some of the computations were repeated with $\xi^2=6.25$ and 11.11 F^{-2}. The effect of changing ξ^2 from 4.0 to 6.25 F^{-2} is typified by comparison of sets number 5 and number 6 in Tables III and IV. With higher values of ξ^2 the high-energy S-wave phase shift becomes too repulsive and the high-energy D-wave phase shift too attractive.

In addition to the results obtained here with five free parameters, Tables III and IV also list the results of the Gammel-Thaler[1] hard-core calculation (3 free parameters) and the results of one of the velocity-dependent models (with four free parameters) used by Razavy, Field, and Levinger.[4] Even with the consideration that the potential shapes used in the present work were more crude than the shapes used by these other authors, the fits obtained here are not as good as the previous fits.

Taken as a whole, however, the results of this section are not discouraging. Further calculations using more sophisticated shapes for $U(r)$ as well as other forms of nonlocal kernels are in progress. If these calculations show that a fit to the singlet-even scattering data may indeed be obtained, then an extension of the model to the other two-body phase shifts and, if justified, eventual application to the many-nucleon problem will be carried out.

ACKNOWLEDGMENTS

The author wishes to express his thanks to Professor R. E. Peierls, Dr. G. Field, and Dr. J. Hetherington for helpful discussions of this problem.

APPENDIX

For mathematical completeness an abbreviated proof that Eq. (4.2) alone ensures that the S-wave scattering length and effective range for the model used in Sec. IV are the same as they would be for $N=0$ is given below. It should be noted that the result does *not* depend on a square shape for $U(r)$.

Since $U(r)=V(r)$, it follows from Eq. (3.4) that $N=0$ implies that the core radius c plays no role; i.e., the core radius cannot appear on the right-hand side of Eq. (3.29). But this parameter appears only in C_l as given in Eqs. (3.32) through (3.34). With $N=0$ then the S-wave phase shift is given by Eq. (3.29) for $l=0$ with $C_0=0$. Furthermore, the functions on the right-hand side of Eq. (3.29) may each be expanded in a Taylor series in k^2. In particular

$$C_0=C_{00}+C_{01}k^2+\cdots, \tag{A1}$$

where C_{00}, C_{01} are C_0 and its derivative with respect to k^2, respectively, evaluated at $k^2=0$. To make the right-hand side of Eq. (3.29) have the form it would have if N vanished it is necessary that

$$C_{00}=0 \quad \text{and} \quad C_{01}=0. \tag{A2}$$

In evaluating these coefficients it follows from $U(r)=V(r)$ and Eqs. (3.15), (3.19), (3.25), and (3.34) that

$$\tilde{\Delta}_l=\Delta_l. \tag{A3}$$

Consider first the condition $C_{00}=0$. For small k^2 Eq. (3.22) yields

$$\beta_2^2=-\beta_0^2+N\beta_0^{-2}k^2-N\xi^2\beta_0^{-6}k^4+\cdots, \tag{A4}$$

$$\beta_1^2=\xi^2\beta_0^{-2}k^2+N\xi^2\beta_0^{-6}k^4+\cdots, \tag{A5}$$

where $\beta_0^2=\xi^2+N>0$. Equations (A3) through (A5) along with Eqs. (3.32) through (3.34) reduce the condition $C_{00}=0$ to the form

$$\Delta_0(1,3;0-\xi^2;c)\Delta_0(1,1;-\beta_0^2,0;c)=0. \tag{A6}$$

The vanishing of the factor $\Delta_0(1,1;-\beta_0^2,0;c)$ in Eq. (A6) requires

$$[\theta_{01}'(0,c)/\theta_{01}(0,c)]=[\theta_{01}'(-\beta_0^2,c)/\theta_{01}(-\beta_0^2,c)], \tag{A7}$$

where $\theta_{01}(\lambda^2,r)=r\phi_{01}(\lambda^2,r)$. From Eq. (3.19) and the fact that there is no singlet-even S-wave bound state, it follows that for $r \leq b$ the sign of the second radial derivative of $\theta_{01}(0,r)$ is the opposite of the sign of this function itself, so that the left-hand side of Eq. (A7) is

negative. But Eq. (3.19) and $\beta_0^2-U(r)>0$ require that the right-hand side of Eq. (A7) be positive. In other words, for Eq. (A6) to be satisfied Eq. (4.2) must hold. Furthermore, by an argument similar to the one just used it may be shown that Eq. (4.2) cannot hold if $\phi_{03}(-\xi^2,r)$ has the behavior given in Eq. (4.3).

By a rather lengthy but straightforward application of the definitions of the functions involved, the other condition implied by Eq. (A2) may be reduced to

$$\Delta_0(1,3;0',-\xi^2;c)\Delta_0(1,1;-\beta_0^2,0;c) -\Delta_0(1,3;-\beta_0^2,-\xi^2;c)\Delta_0(1,1;0',0;c)=0, \quad \text{(A8)}$$

where

$$\Delta_0(1,j;0',\lambda^2;c)=\Delta_0(1,j;0,\lambda^2,c)$$

but with the replacements

$$\phi_{01}(0,c)\rightarrow\frac{d}{d\lambda^2}\phi_{01}(\lambda^2,c)\Big|_{\lambda^2=0},$$

$$\phi_{01}'(0,c)\rightarrow\frac{d}{d\lambda^2}\phi_{01}'(\lambda^2,c)\Big|_{\lambda^2=0}.$$

The left-hand side of Eq. (A9), however, is identically equal to

$$\Delta_0(1,3;0,-\xi^2,c)\Delta_0(1,1;0',-\beta_0^2;c), \quad \text{(A9)}$$

an expression which is necessarily zero by virtue of Eq. (4.2). Therefore, no condition on the parameters other than Eq. (4.2) is needed.

PHYSICAL REVIEW VOLUME 136, NUMBER 6B 21 DECEMBER 1964

Protons from the Deuteron Bombardment of Helium-4

G. G. OHLSEN AND P. G. YOUNG
Research School of Physical Sciences, Australian National University, Canberra
(Received 11 June 1964)

Proton spectra from the bombardment of He^4 with deuterons of energies between 7.7 and 11 MeV have been obtained. A binding energy of -0.93 ± 0.07 MeV and a width of 0.57 ± 0.02 MeV are obtained for the He^5 ground state. A $He^4(d,p)He^5$ excitation function over the Li^6 excitation-energy range from 6.5 to 8.7 MeV failed to confirm the existence of the proposed 7.4-MeV, $T=0$ level. An extrapolation and integration of the present measurements yields an estimate for the total reaction cross section of 460 ± 80 mb at 10 MeV and 460 ± 50 mb at 11 MeV.

I. INTRODUCTION

THIS paper reports on a study of proton spectra from the bombardment of He^4 with deuterons of energies between 7.7 and 11 MeV. Protons may arise through any of the following three reactions or sequences of reactions:

$$\begin{aligned} &d+He^4\rightarrow He^5+p+Q_1,\\ &He^5\rightarrow He^4+n+Q_2, \end{aligned} \quad (1)$$

$$\begin{aligned} &d+He^4\rightarrow Li^5+n+Q_1',\\ &Li^5\rightarrow He^4+p+Q_2', \end{aligned} \quad (2)$$

$$d+He^4\rightarrow He^4+n+p-2.226\text{ MeV}, \quad (3)$$

where He^5 (or Li^5) may be left in its ground state or in an excited state, and where $Q_1+Q_2=Q_1'+Q_2'=-2.226$ MeV.

A study of these reactions is of interest from several points of view. Firstly, from reaction (1), one may obtain information about the width and binding energy of the ground state of He^5; one may also look for evidence for excited states of He^5, although the contributions from reactions (2) and (3) are likely to obscure such effects in a noncoincidence experiment such as the present one.[1] Secondly, the present range of bombarding energies covers the range of Li^6 excitation from 6.5 to 8.7 MeV. If reaction (1) proceeds at least partly via a compound-nucleus mechanism, an excitation function might be expected to yield information about the proposed $T=0$ state in Li^6 near 7.4 MeV.[2] In particular, Sokolov *et al.*[3] reported that this state decays preferentially to He^4+n+p rather than to He^4+d, so the present reaction (1) might be expected to be particularly sensitive to the effects of such a state. Finally, values of the total reaction cross section for deuterons on He^4 may be obtained by extrapolation and integration of the proton spectra obtained in the present measurements; these quantities are useful for a phase-shift or optical-model analysis of the d-He^4 interaction.

Protons from these reactions have been studied previously by several authors. Burge *et al.*[4] observed

[1] For a summary of experiments which bear on the first excited state of He^5 see P. Fessenden and D. R. Maxson, Phys. Rev. **133**, B71 (1964).

[2] F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959).

[3] I. L. Sokolov, M. M. Sulkovskaia, E. I. Karpushkina, and E. A. Albitskaia, Zh. Eksperim. i Teor. Fiz. **30**, 1007 (1956) [English transl.: Soviet Phys.—JETP **3**, 740 (1956)].

[4] E. J. Burge, H. B. Burrows, W. M. Gibson, and J. Rotblat, Proc. Roy. Soc. (London) **A210**, 534 (1951).